Unified algebraic Bethe ansatz for two-dimensional lattice models

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We develop a unified formulation of the quantum inverse scattering method for lattice vertex models associated to the nonexceptional $A^{(2)}_{2r}$, $A^{(2)}_{2r-1}$, $B^{(1)}_{r}$, $C^{(1)}_{r}$, $D^{(1)}_{r+1}$, and $D^{(2)}_{r+1}$ Lie algebras. We recast the Yang-Baxter algebra in terms of different commutation relations between creation, annihilation, and diagonal fields. The solution of the $D^{(2)}_{r+1}$ model is based on an interesting 16-vertex model, which is solvable without recourse to a Bethe ansatz. [S1063-651X(99)07106-8]

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One of the main branches of theoretical and mathematical physics is the theory of exactly solvable models. The most successful approach to construct integrable two-dimensional lattice models of statistical mechanics is by solving the celebrated Yang-Baxter equation [1]. Given a solution of this equation, depending on a continuous parameter λ, one can define the local Boltzmann weights of a commuting family of transfer matrices $T(\lambda)$. A complete understanding of these models should of course, include the exact diagonalization of the transfer matrices, which can provide us with nonperturbative information about the on-shell physical properties such as free-energy thermodynamics and quasiparticle excitation behavior.

The structure of the solutions of the Yang-Baxter equation based on simple Lie algebras is by now fairly well understood [2]. In particular, explicit expressions for the R-matrices related to nonexceptional affine Lie algebras were exhibited in Ref. [3]. Since then, many other R matrices associated to higher dimensional representations of these algebras have also been determined [4]. A long-standing open problem in this field, except for the $A^{(1)}_{r}$ algebra [5], is the diagonalization of their transfer matrices by a first principle approach, i.e., through the quantum inverse scattering method [6,7]. This technique gives us information on the nature of the eigenvectors, which is crucial in the investigation of the off-shell properties such as correlators of physically relevant operators [8]. This fact becomes even more clear thanks to new, recent developments in the calculation of form-factors for integrable models in a finite lattice [9,10].

In this work, we offer the basic tools to solve the remaining vertex models based on the nonexceptional Lie algebras within the quantum inverse scattering framework. Specifically, we present a universal formula for the eigenvectors in terms of the creation fields and the fundamental R-matrix elements of the $A^{(2)}_{2r}$, $A^{(2)}_{2r+1}$, $B^{(1)}_{r}$, $C^{(1)}_{r}$, $D^{(1)}_{r+1}$, and $D^{(2)}_{r+1}$ models. This general construction extends previous work by the author and Ramos [11] and it is crucial in order to accommodate the solution of the twisted $D^{(2)}_{r+1}$ model. It turns out that this solution still depends on the diagonalization of a 16-vertex model having fine-tuned Boltzmann weights. Interesting enough, this latter problem is resolved without recourse to a lattice Bethe ansatz.

One basic object in the quantum inverse scattering method is the monodromy operator $\mathcal{T}(\lambda)$ whose trace over an auxiliary space $\mathcal{A}$ gives us the transfer matrix, $T(\lambda)=Tr_{\mathcal{A}}[\mathcal{T}(\lambda)]$. A sufficient condition for integrability is the existence of an invertible matrix $R(\lambda,\mu)$ satisfying the following relation

$$R(\lambda,\mu)\mathcal{T}(\lambda)\otimes\mathcal{T}(\mu)=\mathcal{T}(\mu)\otimes\mathcal{T}(\lambda)R(\lambda,\mu), \qquad (1)$$

where the matrix elements $R^{\beta_1,\beta_2}_{\alpha_1,\alpha_2}(\lambda,\mu)$ of the R matrix defined on the tensor space $\mathcal{A}\otimes\mathcal{A}$ are c numbers. For the models we are going to discuss in this paper, the R matrix depends only on the difference of the rapidities λ and μ.

As a first step in this program, one may try to construct from the intertwining relation (1) convenient commutation rules for the matrix elements of the monodromy matrix, which in turn can inspire us about the physical content of such elements. There is no known recipe to perform this task, but it certainly begins with an appropriate representation for $\mathcal{T}(\lambda)$ itself. An important input is the reference state $|0\rangle$ one uses to build up the eigenvectors of the transfer matrix $T(\lambda)$. If we choose $|0\rangle$ as the highest weight state for these algebras we soon realize, from the properties of $\mathcal{T}(\lambda)|0\rangle$, that a promising ansatz for the monodromy should be [11]

$$\mathcal{T}(\lambda)=\begin{pmatrix} B(\lambda) & \vec{B}(\lambda) & F(\lambda) \\ \vec{C}^{*}(\lambda) & \hat{A}(\lambda) & \vec{B}^{*}(\lambda) \\ C(\lambda) & \vec{C}(\lambda) & D(\lambda) \end{pmatrix}. \qquad (2)$$

Here the vector $\vec{B}(\lambda)$ and the scalar field $F(\lambda)$ will play the role of creation operators with respect to the reference state $|0\rangle$. The field $\vec{B}(\lambda)$ is a $(q-2)$-component *row* vector, where q is the number of states per bond of these vertex models in a square lattice. Its relation to the rank of each nonexceptional Lie algebra discussed in this paper is given in Table I. The operator $\vec{B}^{*}(\lambda)$ represents $(q-2)$-component *column* vector operator, playing the role as a redundant creation field, and therefore does not enter in our construction of the eigenvectors. The scalar field $\mathbf{C}(\lambda)$, the column and row vectors $\vec{C}^{*}(\lambda)$ and $\vec{C}(\lambda)$ are annihilation operators, $\hat{A}(\lambda)$ is a $(q-2)\times(q-2)$ matrix operator, while the remaining fields $B(\lambda)$ and $D(\lambda)$ are diagonal scalar operators. Putting them together, we have a rather specific $q\times q$ matrix representation for the monodromy matrix.

TABLE I. Parameters of the vertex models associated with the affine Lie algebras. The symbols IK and FZ stand for Izergin-Korepin [15] and Fateev-Zamolodchikov models [14], respectively.

Lie algebra	q	$\hat{r}^{(1)}$ matrix
$A_{2r}^{(2)}$	$2r+1$	19-vertex IK model
$A_{2r-1}^{(2)}$ and $C_r^{(1)}$	$2r$	six-vertex model
$B_r^{(1)}$	$2r+1$	19-vertex FZ model
$D_{r+1}^{(1)}$	$2r+2$	two decoupled six-vertex models
$D_{r+1}^{(2)}$	$2r+2$	16-vertex model

Taking into account this discussion and following the steps of Ref. [11] one can find the appropriate set of fundamental commutation rules between the creation, annihilation, and diagonal fields. However, in order to accommodate the solution of the $D_{r+1}^{(2)}$ vertex model, we lead to generalized expressions for the commutation rules as compared to those exhibited in Ref. [11]. For the sake of simplicity we illustrate these modifications only in the simplest case. This turns out to be the commutation rule between the fields $B(\lambda)$ and $\vec{B}(\lambda)$, which is given by

$$B(\lambda)\vec{B}(\mu)=w_1(\mu-\lambda)\vec{B}(\mu)B(\lambda)-\hat{\eta}(\mu-\lambda)\cdot\vec{B}(\lambda)B(\mu). \tag{3}$$

For the D_{r+1}^2 model $\hat{\eta}(\lambda,\mu)$ is the following matrix Boltzmann weight:

$$\hat{\eta}(x)=\begin{pmatrix} \hat{I}w_2(x) & 0 & 0 & 0 \\ 0 & w_3^-(x) & w_3^+(x) & 0 \\ 0 & w_3^+(x) & w_3^-(x) & 0 \\ 0 & 0 & 0 & \hat{I}w_2(x) \end{pmatrix}, \tag{4}$$

while for the other nonexceptional Lie algebras is just a scalar $\hat{\eta}(x)=w_2(x)$. The operator $\hat{I}$ denotes the $(q/2-2)\times(q/2-2)$ identity matrix, and the expressions for the weights $w_i(x)$ are

$$w_1(x)=\frac{\exp(\alpha x)-k^2}{k[\exp(\alpha x)-1]},$$

$$w_2(x)=\frac{1-k^2}{k[\exp(\alpha x)-1]}, \tag{5}$$

$$w_3^{\pm}(x)=\frac{k^2-1}{2k[1\pm\exp(\alpha x/2)]},$$

where $\alpha=2$ for the $D_{r+1}^{(2)}$ model and $\alpha=1$ for the other nonexceptional Lie algebras listed in Table I. The parameter k describes the ''quantum'' deformation as defined by Jimbo [3]. We remark that many other commutation rules need similar modifications, but since they are sufficiently cumbersome we shall present these technical details elsewhere [12].

We next turn to the analysis of the structure of the eigenvectors. These are multiparticle states characterized by a set of rapidities that parametrize the creation fields, and can be written as linear combination of products of the operators $\vec{B}(\lambda)$ and $F(\lambda)$ acting on the reference state $|0\rangle$. The following physical picture helps us to construct an educated ansatz for such multiparticle states. The field $\vec{B}(\lambda)$ plays the role of a single particle excitation, while $F(\lambda)$ describes a pair excitation, both with bare momenta parametrized by λ. Furthermore, the total number of particles is a conserved quantity thanks to an underlying $U(1)$ invariance. Consequently, $\vec{B}(\lambda_1)$ will represent the one-particle state, the linear combination $\vec{B}(\lambda_1)\otimes\vec{B}(\lambda_2)+\vec{v}(\lambda_1,\lambda_2)F(\lambda_1)$ will be the two-particle state for some unknown vector $\vec{v}(\lambda_1,\lambda_2)$, and so forth. Adapting the steps of Ref. [11] to include the $D_{r+1}^{(2)}$ structure, we find after a tedious computation that the n-particle eigenvector $|\Phi_n(\lambda_1,\ldots,\lambda_n)\rangle$ can be written by the linear combination

$$|\Phi_n(\lambda_1,\ldots,\lambda_n)\rangle=\vec{\Phi}_n(\lambda_1,\ldots,\lambda_n)\cdot\vec{\mathcal{F}}|0\rangle, \tag{6}$$

where the $(q-2)^n$ components of the vector $\vec{\mathcal{F}}$ describe the linear combination and the vector $\vec{\Phi}_n(\lambda_1,\ldots,\lambda_n)$ satisfies the following recurrence relation:

$$\begin{aligned}\vec{\Phi}_n(\lambda_1,\ldots,\lambda_n)&=\vec{B}(\lambda_1)\otimes\vec{\Phi}_{n-1}(\lambda_2,\ldots,\lambda_n)\\ &-\sum_{j=2}^{n}\vec{\xi}(\lambda_1-\lambda_j)\prod_{k=2,k\neq j}^{n}w_1(\lambda_k-\lambda_j)F(\lambda_1)\\ &\otimes\vec{\Phi}_{n-2}(\lambda_2,\ldots,\lambda_{j-1},\lambda_{j+1},\ldots,\lambda_n)B(\lambda_j)\\ &\times\prod_{k=2}^{j-1}\hat{r}_{k,k+1}(\lambda_k-\lambda_j).\end{aligned} \tag{7}$$

In this formula, the vector $\vec{\xi}(x)$ plays the role of a generalized exclusion principle, projecting out certain forbidden states that were made by the creation fields $\vec{B}(\lambda_i)$ from the linear combination. This exclusion rule is governed by the non-null components of this vector, which have been determined in terms of the original R-matrix elements by

$$\vec{\xi}(x)=\sum_{i,j=1}^{q-2}\frac{R_{1q}^{i+1,j+1}(x)}{R_{1q}^{q1}(x)}\hat{e}_i\otimes\hat{e}_j, \tag{8}$$

where $\hat{e}_i$ denotes the elementary projection on the ith position.

The meaning of the auxiliary R-matrix $\hat{r}(x)$ in expression (7) is that it dictates the symmetry of the eigenvectors under permutation of rapidities, namely,

$$\begin{aligned}&\vec{\Phi}_n(\lambda_1,\ldots,\lambda_j,\lambda_{j+1},\ldots,\lambda_n)\\ &=\vec{\Phi}_n(\lambda_1,\ldots,\lambda_{j+1},\lambda_j,\ldots,\lambda_n)\cdot\hat{r}_{j,j+1}(\lambda_j-\lambda_{j+1}).\end{aligned} \tag{9}$$

As long as $r>2$, the structure of the matrix $\hat{r}(x)$ is based on the same Lie algebra as the original R matrix we started with, but now having a lower level rank $r-1$. This structure, however, can change drastically when we reach the lowest level, and for this reason it is convenient to call them separately by $\hat{r}^{(1)}(x)$ matrices. In Table I we describe the type of vertex models that are associated to these $\hat{r}^{(1)}$ matrices for each nonexceptional Lie algebra. We note that for most of the models the underlying $\hat{r}^{(1)}$ matrices are based on the well known six- [13,6] and 19-vertex models [14,15]. The

$D^{(2)}_{r+1}$ model is, however, an exception, and its fundamental $\hat{r}^{(1)}$ matrix is given by a rather peculiar 16-vertex model. Since this result seems to be absent in the literature, we shall present here details about this theory, beginning with its matrix form

$$\hat{r}^{(1)}_{D_2^2}(x)=\begin{pmatrix} a_+(x,k) & -b(k) & -b(k) & c(k)\\ b(k) & -c(k) & a_-(x,k) & b(k)\\ b(k) & a_-(x,k) & -c(k) & b(k)\\ c(k) & -b(k) & -b(k) & a_+(x,k)\end{pmatrix}, \tag{10}$$

where the Boltzmann weights $a_\pm(x,k)$, $b(k)$, and $c(k)$ are given by

$$a_\pm(x,k)=f_\pm(k)+g(x), \quad c(k)=\frac{2b^2(k)}{f_+(k)-f_-(k)}. \tag{11}$$

This matrix is factorizable for arbitrary functions $f_\pm(k)$ and $g(x)$, but in the specific case of the $D^{(2)}_{r+1}$ model we have the following extra constraints:

$$g(x)=\sinh(x), \quad f_+(k)=-f_-(k),$$
$$f_+(k)=c(k)\pm(k-1/k)/2. \tag{12}$$

To make further progress for the algebraic Bethe ansatz solution of the $D^{(2)}_{r+1}$ it is necessary to diagonalize the auxiliary transfer matrix associated to the $\hat{r}^{(1)}$-matrix (10) in the presence of inhomogeneities. More precisely, we have to tackle the following eigenvalue problem:

$$\hat{r}^{(1)}(\lambda-\mu_1)^{c_1a_1}_{b_1d_1}\hat{r}^{(1)}(\lambda-\mu_2)^{d_1a_2}_{b_2c_2}\cdots\hat{r}^{(1)}$$
$$\times(\lambda-\mu_n)^{d_{n-1}a_n}_{b_nc_1}\mathcal{F}^{a_n\cdots a_1}=\Lambda^{(1)}_{D_2^2}(\lambda,\{\mu_j\})\mathcal{F}^{b_n\cdots b_1}, \tag{13}$$

where $\{\mu_j\}$ stands for the inhomogeneities.

We solve this problem by first mapping the 16-vertex model to an asymmetric eight-vertex model, following a procedure devised long ago by Wu [16]. This leads us to a much simpler vertex model, having the following R matrix:

$$\hat{r}^{(1)}_{8v}(x)=\begin{pmatrix} \tilde{a}(x) & 0 & 0 & \tilde{d}_+(k)\\ 0 & \tilde{c}(k) & \tilde{a}(x) & 0\\ 0 & \tilde{a}(k) & \tilde{c}(k) & 0\\ \tilde{d}_-(k) & 0 & 0 & \tilde{a}(x)\end{pmatrix}, \tag{14}$$

where the respective weights are given by

$$\tilde{a}(x)=\sinh(x), \quad \tilde{c}(k)=f_+(k)-b^2(k)/f_+(k),$$
$$\tilde{d}_\pm(k)=f_+(k)+b^2(k)/f_+(k)\pm 2b(k). \tag{15}$$

The Boltzmann weights of this eight-vertex model have enough special properties to allow us exact diagonalization without the need of a Bethe ansatz analysis. In fact, the off-diagonal matrix component of the corresponding Lax operator commutes due to the relation $d_+(k)d_-(k)=c^2(k)$, which is also a restriction for factorization of the r-matrix (14). This leads us to conclude that all the eigenvectors are given in terms of products of on site states, and its expression in the spin-$\frac{1}{2}\sigma^z$ basis is

$$\vec{\mathcal{F}}_{8v}=\prod_{\epsilon_i=\pm1}\prod_{i=1}^{m}\otimes\begin{pmatrix}1\\ \epsilon_i\sqrt{\dfrac{\tilde{c}(k)}{\tilde{d}_+(k)}}\end{pmatrix}, \tag{16}$$

where $\epsilon_i=\pm1$ are Z_2 variables, parametrizing the many possible $(2)^m$ states. The expression for the eigenstates of the original 16-vertex model can be obtained from Eq. (16) after a transformation to the representation where σ^x is diagonal. It is interesting to note that these are typical variational states, with the advantage of being exact and valid for the role spectrum.

Now the eigenvalues can be determined almost directly, and they are given by

$$\Lambda^{(1)}_{D_2^2}(\lambda,\{\mu_j\})=\prod_{i=1}^{m}[\sinh(\lambda-\mu_i)-\epsilon_i(k-1/k)/2]$$
$$+\prod_{i=1}^{m}[\sinh(\lambda-\mu_i)+\epsilon_i(k-1/k)/2]. \tag{17}$$

These latter results are fundamental in order to solve the eigenvalue problem for the $D^{(2)}_{r+1}$ from first principles. In particular, the fact that the eigenvectors do not depend on the rapidities is an essential feature to match the inhomogeneous eigenvalue and nested Bethe ansatz problems, since the $\hat{r}^{(1)}$ matrix cannot be made regular. It should be also emphasized that this theory is not in the class of the so-called free-fermion models.

In summary, we have developed a framework that is capable of dealing with the transfer matrix eigenvalue problem of the vertex models based on nonexceptional Lie algebras from a unified point of view. Our nested Bethe ansatz results for the eigenvalues and Bethe ansatz equations corroborate those conjectured in Ref. [17] by means of analyticity assumptions. The many technical details of such nested Bethe ansatz analysis will be presented elsewhere [12]. The universal formula we have obtained for the eigenvectors paves the way to a general off-shell Bethe ansatz formulation, and consequently could be useful to produce integral representations for the form factors [9]. Finally, we remark that all the models solved in this work share a common algebraic structure, i.e., the braid-monoid algebra [18,19]. A question that promptly arises is if there are direct connections between the braid-monoid algebra and our algebraic Bethe ansatz framework. In this sense, we note that the size (three) of the matrix representation for $\mathcal{T}(\lambda)$ coincides with the number of eigenvalues of the braid operator [18]. The same observation also works for the Hecke algebra, where the number of eigenvalues is 2 [18]. It remains to be seen whether this is an isolated coincidence or the tip of an iceberg.

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[1] C. N. Yang, Phys. Rev. Lett. **19**, 1312 (1967); R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, New York, 1982).
[2] V. G. Drinfeld, in *Proceedings of ICM Berkeley* (Academic Press, New York, 1986), Vol. 1, p. 798.
[3] M. Jimbo, Commun. Math. Phys. **102**, 247 (1986); V. V. Bazhanov, Phys. Lett. B **159**, 321 (1985).
[4] G. W. Delius, M. D. Gould, and Y. Z. Zhang, Nucl. Phys. B **432**, 377 (1994); Int. J. Mod. Phys. A **11**, 3415 (1996), and references therein.
[5] O. Babelon, H. J. de Vega, and C. M. Viallet, Nucl. Phys. B **200**, 266 (1982) P. P. Kulish and N. Yu. Reshetikhin, Zh. Eksp. Teor. Fiz. **80**, 109 (1981) [Sov. Phys. JETP **54**, 108 (1981)].
[6] L. D. Faddeev, Sov. Sci. Rev. **C1**, 107 (1980); L. A. Takhtajan and L. D. Faddeev, Russ. Math. Sur. **34**, 11 (1979); L. A. Takhtajan, in *Exactly Solvable Problems in Condensed Matter and Relativistic Field Theory*, edited by B. S. Shastry *et al.*, Lectures Notes in Physics Vol. 242 (Springer-Verlag, Berlin, 1985), p. 175.
[7] H. B. Thacker, Rev. Mod. Phys. **53**, 253 (1981); H. J. de Vega, Int. J. Mod. Phys. A **4**, 2371 (1989).
[8] V. E. Korepin, G. Izergin, and N. M. Bogoliubov, *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge University Press, Cambridge, 1992).
[9] H. Babujian, M. Karowski, and A. Zapletal, J. Phys. A **30**, 6425 (1997); H. Babujian, A. Fring, M. Karowski, and A. Zapletal, e-print hep-th/9805185.
[10] N. Kitanine, J. M. Maillet, and V. Terras, e-print math-ph/9807020.
[11] M. J. Martins and P. B. Ramos, Nucl. Phys. B **500**, 579 (1997); **522**, 413 (1998).
[12] M. J. Martins (unpublished).
[13] E. H. Lieb, Phys. Rev. Lett. **18**, 692 (1967).
[14] A. B. Zamolodchikov and V. Fattev, Sov. J. Nucl. Phys. **32**, 298 (1980).
[15] A. G. Izergin and V. E. Korepin, Commun. Math. Phys. **79**, 303 (1981).
[16] F. Y. Wu, Solid State Commun. **10**, 115 (1972).
[17] N. Yu. Reshetikhin, Lett. Math. Phys. **14**, 125 (1987); H. J. de Vega and E. Lopes, Nucl. Phys. B **362**, 261 (1991).
[18] V. F. R. Jones, Int. J. Mod. Phys. B **4**, 701 (1990).
[19] M. Wadati, T. Deguchi, and Y. Akutsu, Phys. Rep. **180**, 247 (1989); U. Grimm, J. Phys. A **27**, 5897 (1994); Lett. Math. Phys. **32**, 183 (1994).